A Continuous Time Approach for the Asymptotic Value in Two-Person Zero-Sum Repeated Games

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- All sets under consideration are finite.

Inductively, at stage $t = 1, ...,$ knowing the past history $h_t = (\omega_1, i_1, j_1, ..., i_{t-1}, j_{t-1}, \omega_t)$, player I chooses $i_t \in I$, player J chooses $j_t \in J$.

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This discounted game has a value v_λ.

The Shapley Operator

The Shapley operator $T(\lambda, \cdot)$ associates to a function f in $\mathbb{R}^\Omega$ the function:

$$T(\lambda, f)(\omega) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} [\lambda g(x, y, \omega) + (1 - \lambda) \sum_{\tilde{\omega}} Q(x, y, \omega)(\tilde{\omega}) f(\tilde{\omega})]$$

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Lemma

The Shapley operator $T(\lambda, \cdot)$ is well defined from $\mathbb{R}^{\Omega}$ to itself. Its unique fixed point is v_{λ}.

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Given the parameter $m = (k, \ell)$, each player knows his own component and holds a prior on the other player's component.
From stage 1 on, the parameter is fixed and the information of the players after stage n is $a_{n+1} = b_{n+1} = \{i_n, j_n\}$.

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$\mathbf{X} = \Delta(I)^K$ and $\mathbf{Y} = \Delta(J)^L$ are the type-dependent mixed action sets of the players; g is extended on $\mathbf{X} \times \mathbf{Y} \times M'$ by
$g(p, q, x, y) = \sum_{k,\ell} p^k q^\ell g(k, \ell, x^k, y^\ell)$.

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$g(p, q, x, y) = \sum_{k,\ell} p^k q^\ell g(k, \ell, x^k, y^\ell)$.
Given (p, q, x, y), let $x(i) = \sum_k x_i^k p^k$ be the (total) probability of action i and $p(i)$ be the conditional probability on K given the action i, explicitly $p^k(i) = \frac{p^k x_i^k}{x(i)}$ (and similarly for y and q).

The resulting form of the Shapley operator is:

$$\begin{aligned} T(\lambda, f)(p,q) &= \sup_{x \in \mathbf{X}} \inf_{y \in \mathbf{Y}} \{\lambda \sum_{k,\ell} p^k q^\ell g(k,\ell,x^k,y^\ell) \\ &\quad + (1-\lambda) \sum_{i,j} x(i)y(j) f(p(i),q(j))\} \end{aligned} \tag{1}$$

These equations are due to Aumann and Maschler (1966) and Mertens and Zamir (1971).

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and the recursive formula for the n stage value is obtained similarly

$$v_n = \mathbf{T}[\frac{1}{n}, v_{n-1}] \tag{3}$$

with obviously $v_0 = 0$.

Consider now an arbitrarly evaluation probability μ on $\mathbb{N}^\star$. The total payoff is $\sum_t \mu_m g_m$.

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The repeated game is naturally represented as a game played between times 0 and 1, where the actions are constant on each subinterval (t_{k-1}, t_k): its length μ_k is the weight of stage k in the original game. Let v_Π be its value.

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$$v_\Pi = \mathtt{val}\{t_1 g_1 + (1 - t_1)\mathsf{E} v_{\Pi_{t_1}}\}$$

where Π_{t_1} is the normalization on $[0,1]$ of the trace of the partition Π on the interval $[t_1, 1]$.

Define now $V_\Pi(t_k)$ as the value of the game starting at time t_k with evaluation $\sum_m \mu_{m+k} g_m$. One obtains the alternative recursive formula

$$V_\Pi(t_k) = \mathtt{val}\{(t_{k+1} - t_k) g_{k+1} + \mathsf{E} V_\Pi(t_{k+1})\} \tag{4}$$

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By taking the linear extension we define this way for every finite partition Π, a function $V_\Pi(t)$ on $[0, 1]$.

Lemma

Assume $\mu(n)$ decreasing. Then V_Π is C-Lipschitz in t, where C is a bound on the payoffs.

Contents

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1) Concerning games with incomplete information on one side, the first results proving the existence of $\lim_{n\to\infty} v_n$ and $\lim_{\lambda\to 0} v_\lambda$ are due to Aumann and Maschler (1966), including in addition an identification of the limit as $\mathtt{Cav}_{\Delta(K)} u$.

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Here $u(p) = \mathtt{val}_{\Delta(I)\times\Delta(J)} \sum_k p^k g(k,x,y)$ is the value of the one shot non revealing game, where the informed player does not use his information and $\mathtt{Cav}_C$ is the concavification operator: given ϕ, a real bounded function defined on a convex set C, $\mathtt{Cav}_C(\phi)$ is the smallest function greater than ϕ and concave, on C.

Extensions of these results to games with lack of information on both sides were achieved by Mertens and Zamir (1971). In addition they identified the limit as the only solution of the system of implicit functional equations with unknown ϕ:

$$\phi(p, q) = \mathtt{Cav}_{p \in \Delta(K)} \min\{\phi, u\}(p, q), \tag{5}$$

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Here again u stands for the value of the non revealing game: $u(p,q) = \mathtt{val}_{X \times Y} \sum_{k,\ell} p^k q^\ell g(k,\ell,x,y)$ and we will write **MZ** for the corresponding operator

$$\phi = \mathbf{MZ}(u). \tag{7}$$

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the Shapley equation can be written as a finite set of polynomial equalities and inequalities involving $\{x^k_\lambda, y^k_\lambda, v_\lambda(k), \lambda\}$ thus it defines a semi-algebraic set in some euclidean space $\mathbb{R}^N$, hence by projection v_λ has an expansion in Puiseux series.

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The existence of $\lim_{n \to \infty} v_n$ is obtained by an algebraic comparison argument, Bewley and Kohlberg (1976).

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We describe here an approach that was initially introduced by Laraki (2002) for the discounted case.

Contents

The analysis is inspired from Larki (2001).
Recall that the recursive equation is a fixed point operator:

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\begin{aligned}
\mathbf{T}(\lambda, v_\lambda)(p,q) &= \sup_{x \in \mathbf{X}} \inf_{y \in \mathbf{Y}} \{\lambda g(p,q,x,y) \\
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Remark that the family of functions $\{v_\lambda(p,q)\}$ is uniformly Lipschitz, hence relatively compact. To prove convergence it is enough to show that there is only one accumulation point.

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Remark that the family of functions $\{v_\lambda(p,q)\}$ is uniformly Lipschitz, hence relatively compact. To prove convergence it is enough to show that there is only one accumulation point.
Note first that any accumulation point w satisfies

$$\mathbf{T}(0,w) = w \tag{9}$$

i.e. is a fixed point of the projective operator $T(0,.)$.

Assume now that w_1 and w_2, $w_1 \geq w_2$ are two different accumulation points and let (p_0, q_0) an extreme point of the (convex hull of) the set where the difference $w_1 - w_2$ is maximal.

Assume now that w_1 and w_2, $w_1 \geq w_2$ are two different accumulation points and let (p_0, q_0) an extreme point of the (convex hull of) the set where the difference $w_1 - w_2$ is maximal. Using (9) and the fact that all functions involved are saddle, this implies that the set $\mathbf{X}(0, w_1)(p_0, q_0)$ of profile of mixed actions $x \in \mathbf{X}$ optimal in $\mathbf{T}(0, w_1)$ is included in the set $NR_X(p_0)$ of non revealing actions at p_0 (meaning that for any move i having positive probability $p(i) = p_0$).

Consider a sequence v_{λ_n} converging to w_1 and let x_n be optimal for $\mathbf{T}(\lambda_n, v_{\lambda_n})(p_0, q_0)$. Jensen's inequality leads to

$$v_{\lambda_n}(p_0, q_0) \leq \lambda_n g(p_0, q_0, x_n, y) + (1 - \lambda_n) v_{\lambda_n}(p_0, q_0) \qquad \forall y \in Y$$

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$$w_1(p_0, q_0) \leq g(p_0, q_0, \bar{x}, y) \qquad \forall y \in Y$$

which implies $w_1(p_0, q_0) \leq u(p_0, q_0)$, since $\bar{x} \in NR_X(p_0)$ (by upper semi continuity of $\mathbf{X}(\lambda_n, v_{\lambda_n})(p_0, q_0)$).

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The dual property implies convergence.

Contents

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For each integer n, let $W_n(1, p, q) := 0$ and for $m = 0, ..., n-1$ define $W_n(\frac{m}{n}, p, q, \omega)$ inductively as follows.

$$
\begin{aligned}
W_n(\frac{m}{n}, p, q) \quad = \quad & \max_x \min_y [\frac{1}{n} g(x, y, p, q) \\
& + \sum_{i,j} \overline{x}(i) \overline{y}(j) W_n(\frac{m+1}{n}, p(i), q(j))]
\end{aligned}
$$

Extend $W_n(\cdot, p, q)$ to $[0, 1]$ by linear interpolation and consequently: $W_n(\cdot, \cdot, \cdot)$ is a C Lipschitz function.

Moreover if W is an accumulation point of the equi-continuous family $\{W_n\}$ then for all (t, p, q):

$$W(t, p, q) = \max_x \min_y \left[\sum_{i,j} \overline{x}(i)\overline{y}(j) W(t, p(i), q(j)) \right]$$

Let $\mathbf{X}(t, p, q, W) \subseteq \Delta(I)^K$ be the set of strategies for player I that are optimal for the above game.

The variational Inequalities

Theorem

For any accumulation point W of the family $\{W_n\}$, all $(p,q) \in \Delta(K) \times \Delta(L)$ and all C^1 test function $\phi : [0,1] \to \mathbf{R}$:

(P1) *If, for some $t \in [0,1)$, $\mathbf{X}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q) - \phi(\cdot)$ has a global maximum at t, then $u(p,q) + \phi'(t) \geq 0$.*

(P2) *If, for some $t \in [0,1)$, $\mathbf{Y}(t,p,q,W)$ is non-revealing and $W(\cdot,p,q) - \phi(\cdot)$ has a global minimum at t then $u(p,q) + \phi'(t) \leq 0$.*

proof

- Let t, p and q such that $\mathbf{X}(t, p, q, W)$ is non-revealing and $W(\cdot, p, q) - \phi(\cdot)$ admits a global maximum at t.

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- Let $W_{\varphi(n)}$ converge to W and define $\theta(n) \in \{0, \ldots, \varphi(n) - 1\}$ such that $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot, p, q) - \phi(\cdot)$. Then $\frac{\theta(n)}{\varphi(n)} \to t$.

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- One has

$$W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)},p,q\right) = \max_x \min_y [\frac{1}{\varphi(n)}g(x,y,p,q) + \sum_{i,j}\overline{x}(i)\overline{y}(j)W_{\varphi(n)}(\frac{\theta(n)+1}{\varphi(n)},p(i),q(j))]$$

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\begin{aligned}
W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)}, p, q\right) \quad &\leq \quad \frac{1}{\varphi(n)} g(x_n, y, p, q) \\
&\quad + \sum_i \overline{x}_n(i) W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)}, p_n(i), q\right)
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Let x_n be optimal for the maximizer and $y \in Y$ be any non-revealing strategy of player J.

$$\begin{aligned} W_{\varphi(n)}\left(\frac{\theta(n)}{\varphi(n)}, p, q\right) &\leq \frac{1}{\varphi(n)} g(x_n, y, p, q) \\ &+ \sum_i \overline{x}_n(i) W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)}, p_n(i), q\right) \end{aligned}$$

By concavity of $W_{\varphi(n)}$ with respect to p

$$\sum_{i \in I} \overline{x}_n(i) W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)}, p_n(i), q\right) \leq W_{\varphi(n)}\left(\frac{\theta(n)+1}{\varphi(n)}, p, q\right) .$$

proof

Hence:

$$0 \leq g(x_n, y, p, q) + \varphi(n) \left[W_{\varphi(n)} \left(\frac{\theta(n)+1}{\varphi(n)}, p, q \right) - W_{\varphi(n)} \left(\frac{\theta(n)}{\varphi(n)}, p, q \right) \right]$$

Since $\frac{\theta(n)}{\varphi(n)}$ is a global maximum of $W_{\varphi(n)}(\cdot, p, q) - \phi(\cdot)$:

$$\phi\left(\frac{\theta(n)+1}{\varphi(n)}\right) - \phi\left(\frac{\theta(n)}{\varphi(n)}\right) \geq W_{\varphi(n)} \left(\frac{\theta(n)+1}{\varphi(n)}, p, q \right) - W_{\varphi(n)} \left(\frac{\theta(n)}{\varphi(n)}, p$$

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Hence:

$$0 \leq g(x_n, y, p, q) + \varphi(n) \left[W_{\varphi(n)} \left(\frac{\theta(n)+1}{\varphi(n)}, p, q \right) - W_{\varphi(n)} \left(\frac{\theta(n)}{\varphi(n)}, p, q \right) \right.$$

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Assume $\{x_n\}$ converges to some x (hence non-revealing).

Passing to the limit:

$$g(x, y, p, q) + \phi'(t) \geq 0 \ .$$

Since this inequality holds true for every y, taking the maximum with respect to x yields:

$$u(p, q) + \phi'(t) \geq 0 \ .$$

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Let W_1 and W_2 be two fixed points of Φ in $\mathcal{G}$ and suppose that:

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- ***(P3)** $W_1(1, p, q) \leq W_2(1, p, q)$ for any $(p, q) \in \Delta(K) \times \Delta(L)$.*

Then $W_1 \leq W_2$ on $[0, 1] \times \Delta(K) \times \Delta(L)$.

The comparison principle

We argue by contradiction, assuming that

$$\max_{t\in[0,1],p\in P,q\in Q}[W_1(t,p,q)-W_2(t,p,q)]=\delta>0\ .$$

Then, for $\varepsilon>0$ sufficiently small,

$$\delta(\varepsilon):=\max_{t\in[0,1],s\in[0,1],p\in P,q\in Q}[W_1(t,p,q)-W_2(s,p,q)-\frac{(t-s)^2}{2\varepsilon}+\varepsilon s]\ >0 \tag{10}$$

Moreover $\delta(\varepsilon)\to\delta$ as $\varepsilon\to 0$.

The comparison principle

We claim that there is $(t_\varepsilon, s_\varepsilon, p_\varepsilon, q_\varepsilon)$, point of maximum above, such that $X(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I and $Y(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J.

The comparison principle

We claim that there is $(t_\varepsilon, s_\varepsilon, p_\varepsilon, q_\varepsilon)$, point of maximum above, such that $X(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I and $Y(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J.
Finally we note that $t_\varepsilon < 1$ and $s_\varepsilon < 1$ for ε sufficiently small, because $\delta(\varepsilon) > 0$ and $W_1(1, p, q) \leq W_2(1, p, q)$ for any (p, q) by **P3**.

The comparison principle

Since the map $t \mapsto W_1(t, p_\varepsilon, q_\varepsilon) - \frac{(t-s_\varepsilon)^2}{2\varepsilon}$ has a global maximum at t_ε and since $X(t_\varepsilon, p_\varepsilon, q_\varepsilon, W_1)$ is non-revealing for player I, condition **P1** implies that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} \geq 0 \,. \tag{11}$$

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In the same way, since the map $s \mapsto W_2(s, p_\varepsilon, q_\varepsilon) + \frac{(t_\varepsilon - s)^2}{2\varepsilon} - \varepsilon s$ has a global minimum at s_ε and since $Y(s_\varepsilon, p_\varepsilon, q_\varepsilon, W_2)$ is non-revealing for player J, we have by condition **P2** that

$$u(p_\varepsilon, q_\varepsilon) + \frac{t_\varepsilon - s_\varepsilon}{\varepsilon} + \varepsilon \leq 0 \, .$$

The comparison principle

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This latter inequality contradicts (11).

Contents

The same tols extend to the study of absorbing games and can be applied to the "splitting game".
Sketch of the approach :
The family of value functions is relatively compact
Consider two accumulation points w_1 and w_2 and a point (t, ω) where the difference $w_1 - w_2$ is maximal.
Deduce a variational inequality at (t, ω) for any majorant of w_1 and a dual property
Prove a comparison principle.